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Formulation of Navier–Stokes Equations for Moving Grid and Boundary

Homayun K. Navaz* and Raymond M. Berg†
Kettering University, Flint, Michigan 48504-4898

Nomenclature

E = total energy
$\boldsymbol{E}, \boldsymbol{F}$ = flux vectors
f = force per unit mass
$\boldsymbol{G}$ = source term vector
H = total enthalpy
J = Jacobian of the transformation
n = 0 for two-dimensional flows, 1 for axisymmetric flows
P = pressure
$\boldsymbol{Q}$ = primitive variable vector
q = heat flux
r = radial direction
T = temperature
t = time
$\boldsymbol{U}$ = conserved variable vector
u, v = components of velocity in x and r directions
W = boundary work per unit mass of the fluid
x = axial direction
γ = specific heat ratio
μ = molecular viscosity
$\xi, \eta, \tilde{t}$ = computational coordinates
ρ = density
τ = shear stress tensor

Subscripts
r = refers to the radial direction
v = viscous flux
x = refers to the axial direction

Superscripts
T = transpose of a matrix
$\cdot$ = rate, change in time
$\sim$ = transformed coordinates

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*Assistant Professor, Department of Mechanical Engineering, 1700 West Third Avenue.

†Lecturer, Department of Mechanical Engineering, 1700 West Third Avenue.

Introduction

In problems where a boundary is moving, part of the control volume energy can be consumed to displace the boundary, e.g., expansion, or the boundary can be utilized to increase the energy content of the control volume, e.g., compression. This effect can be included in the energy equation through a source term. The energy equation in the Navier–Stokes formulation contains the rate of work generated by the flow, and it does not include other types of energy that may influence the control volume such as the boundary work, shaft work, etc. Generally, the computational fluid dynamics (CFD) codes have mostly been applied to open systems with only the flow work being present in the formulation. There are several commercially available programs that are applied to closed systems such as the gun-barrel propulsion and internal combustion (IC) engines. However, these CFD codes with all degrees of sophistication must be able to reproduce good results for simple classic problems.

The transformation of the Navier–Stokes equations into a computational coordinate system is widely practiced in finite difference and finite volume methods.[1,2] This coordinate mapping introduces the transformation Jacobian J, a representation of a cell volume, into the equations. For a time-varying grid, the value of the Jacobian that is calculated for the purpose of evaluating the flow variables must be consistent with the value of the appropriate differential volume element; therefore, requiring a time derivative of the Jacobian to be included in the finite difference form of equations. This time derivative is part of the geometric conservation law (GCL). Thomas and Lombard[3] derived a geometric conservation law having the same integral form as the mass conservation. Gaitonde and Fiddes[4] applied the GCL to the finite volume solution of the Navier–Stokes equation. To the best of our knowledge, the requirements for an energy source/sink term and time derivative of the transformation Jacobian are sometimes ignored by CFD practitioners.

In this paper, we examine the effects of the inconsistency in the computation of the effective volume element associated with moving grids, and the boundary work on the solution of conservation equations, for a simple isentropic expansion problem of a closed system. The liquid thrust chamber performance (LTCP) code,[5] which is based on finite difference discretization with a total variation diminishing (TVD) scheme, with options to be applied to Lax–Friedrichs, Van–Leer, or Roe numerical fluxes, is used as a vehicle for this study.

Theory

The Navier–Stokes equations for compressible flow of gases in a two-dimensional/axisymmetric coordinate system are

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{E}}{\partial x} + \frac{\partial \boldsymbol{E}_v}{\partial x} + \frac{\partial \boldsymbol{F}}{\partial r} + \frac{\partial \boldsymbol{F}_v}{\partial r} + \boldsymbol{G} = 0 \tag{1}$$

where $\boldsymbol{U}$ is the conserved variable vector, $\boldsymbol{E}$ and $\boldsymbol{F}$ are the two components of inviscid flux vectors and $\boldsymbol{E}_v$ and $\boldsymbol{F}_v$ are their viscous counterparts, and $\boldsymbol{G}$ is the source term. These terms are, respectively, defined as

$$\boldsymbol{U} = r^n(\rho u, \rho v, E, \rho)^T \tag{2a}$$

$$\boldsymbol{E} = r^n(\rho u^2 + P, \rho uv, \rho uH, \rho u)^T \tag{2b}$$

$$\boldsymbol{F} = r^n(\rho uv,\ \rho v^2 + P,\ \rho vH,\ \rho v)^T \tag{2c}$$

$$\boldsymbol{E}_v = r^n[\tau_{xx},\ \tau_{xr},\ u\tau_{xx} + v\tau_{xr} + q_x,\ 0]^T \tag{2d}$$

$$\boldsymbol{F}_v = r^n[\tau_{xr},\ \tau_{rr},\ u\tau_{xr} + v\tau_{rr} + q_r,\ 0]^T \tag{2e}$$

$$\boldsymbol{G} = r^n[0,\ nP,\ \dot{W},\ 0]^T \tag{2f}$$

If a moving boundary is present, the inclusion of the power consumed by the moving boundary, $\dot{W}$, in the source term is necessary. The boundary can be moved by calculating the force at each instant of time, and allowing the boundary to move according to Newton's second law. The power sink can be evaluated by

$$\dot{W} = \rho f_x(\Delta x/\Delta t) + \rho f_r(\Delta r/\Delta t) \tag{3}$$

where f is force per unit mass of the gas, and can be obtained by integrating pressure over the surface of the moving boundary. The details of the transformation into the computational coordinate system can be found in Refs. 1 and 2. In the transformed coordinates, the preceding equations can be written as

$$\frac{\partial \tilde{\boldsymbol{U}}}{\partial \tilde{t}} + \frac{\partial \tilde{\boldsymbol{E}}}{\partial \xi} + \frac{\partial \tilde{\boldsymbol{E}}_v}{\partial \xi} + \frac{\partial \tilde{\boldsymbol{F}}}{\partial \eta} + \frac{\partial \tilde{\boldsymbol{F}}_v}{\partial \eta} + \tilde{\boldsymbol{G}} = 0 \tag{4}$$

where

$$\tilde{\boldsymbol{U}} = \frac{r^n}{J}\boldsymbol{U} \tag{5a}$$

$$\tilde{\boldsymbol{E}} = \frac{r^n}{J}[\xi_t \boldsymbol{U} + \xi_x \boldsymbol{E} + \xi_r \boldsymbol{F}]^T, \qquad \tilde{\boldsymbol{E}}_v = \frac{r^n}{J}[\xi_x \boldsymbol{E}_v + \xi_r \boldsymbol{F}_v]^T \tag{5b}$$

$$\tilde{\boldsymbol{F}} = \frac{r^n}{J}[\eta_t \boldsymbol{U} + \eta_x \boldsymbol{E} + \eta_r \boldsymbol{F}]^T, \quad \tilde{\boldsymbol{F}}_v = \frac{r^n}{J}[\eta_x \boldsymbol{E}_v + \eta_r \boldsymbol{F}_v]^T \tag{5c}$$

$$\tilde{\boldsymbol{G}} = \frac{r^n}{J}\boldsymbol{G} \tag{5d}$$

Linearization

The linearization is performed with respect to the primitive variable vector $\boldsymbol{Q}$, which is given by $\boldsymbol{Q} = [u, v, T, P]^T$. Consider the Taylor series expansion of the flux:

$$\tilde{\boldsymbol{E}}^{n+1} = \tilde{\boldsymbol{E}}^n + \frac{\partial \tilde{\boldsymbol{E}}}{\partial \tilde{t}}\Delta \tilde{t} + \mathcal{O}(\Delta \tilde{t})^2 \tag{6}$$

where the superscript n refers to the time increment. However, the flux in the ξ direction in a dynamic grid system is expressed as

$$\tilde{\boldsymbol{E}} = \tilde{\boldsymbol{E}}(\boldsymbol{Q}, \xi_t, \xi_x, \xi_r) \tag{7}$$

The chain rule will yield the following relation for the time derivative of the flux:

$$\frac{\partial \tilde{\boldsymbol{E}}}{\partial \tilde{t}} = \frac{\partial \tilde{\boldsymbol{E}}}{\partial \boldsymbol{Q}}\frac{\partial \boldsymbol{Q}}{\partial \tilde{t}} + \frac{\partial \tilde{\boldsymbol{E}}}{\partial \xi_t}\frac{\partial \xi_t}{\partial \tilde{t}} + \frac{\partial \tilde{\boldsymbol{E}}}{\partial \xi_x}\frac{\partial \xi_x}{\partial \tilde{t}} + \frac{\partial \tilde{\boldsymbol{E}}}{\partial \xi_r}\frac{\partial \xi_r}{\partial \tilde{t}} \tag{8}$$

Inserting Eq. (8) into Eq. (6) yields

$$\tilde{\boldsymbol{E}}^{n+1} = \tilde{\boldsymbol{E}}^n + \frac{\partial \tilde{\boldsymbol{E}}}{\partial \boldsymbol{Q}}\Delta \boldsymbol{Q} + \frac{\partial \tilde{\boldsymbol{E}}}{\partial \xi_t}\Delta \xi_t + \frac{\partial \tilde{\boldsymbol{E}}}{\partial \xi_x}\Delta \xi_x + \frac{\partial \tilde{\boldsymbol{E}}}{\partial \xi_r}\Delta \xi_r \tag{9}$$

For the conserved-variable vector, we may write

$$\frac{\partial \tilde{\boldsymbol{U}}}{\partial \tilde{t}} = \frac{r^n}{J}\Gamma\frac{\partial \boldsymbol{Q}}{\partial \tilde{t}} - \frac{r^n}{J^2}\boldsymbol{U}\frac{\partial J}{\partial \tilde{t}} \tag{10}$$

where $\Gamma = \partial \boldsymbol{U}/\partial \boldsymbol{Q}$.

Numerical Technique

The finite difference method was utilized to discretize the governing equations. The alternating direction implicit (ADI) method was used to split Eq. (4) into two equations.[1] The numerical scheme that was used to solve the equations in the transformed coordinates is based on the monotonic upstream schemes for conservation laws (MUSCL)[6] TVD[7] technique applied to Van–Leer numerical flux splitting:

$$\begin{aligned}\Gamma\Delta^n\boldsymbol{Q} + \Delta\tilde{t}\Bigg\{&\frac{\partial}{\partial \xi}\left[\frac{\partial \tilde{\boldsymbol{E}}}{\partial \boldsymbol{Q}} + \frac{\partial \tilde{\boldsymbol{E}}_v}{\partial \boldsymbol{Q}_\xi} + \frac{\partial \tilde{\boldsymbol{E}}_v}{\partial \boldsymbol{Q}} - \left(\frac{\partial \tilde{\boldsymbol{E}}_v}{\partial \boldsymbol{Q}_\xi}\right)_\xi\right] \\ &+ \frac{\partial}{\partial \eta}\left[\frac{\partial \tilde{\boldsymbol{F}}}{\partial \boldsymbol{Q}} + \frac{\partial \tilde{\boldsymbol{F}}_v}{\partial \boldsymbol{Q}_\eta} + \frac{\partial \tilde{\boldsymbol{F}}_v}{\partial \boldsymbol{Q}} - \left(\frac{\partial \tilde{\boldsymbol{F}}_v}{\partial \boldsymbol{Q}_\eta}\right)_\eta\right] \\ &+ \frac{\partial \tilde{\boldsymbol{G}}}{\partial \boldsymbol{Q}}\Bigg\}\Delta^n\boldsymbol{Q} = \mathrm{RHS}\end{aligned} \tag{11}$$

$$\mathrm{RHS} = -\Delta\tilde{t}\left[\frac{\partial \tilde{\boldsymbol{E}}}{\partial \xi} + \frac{\partial \tilde{\boldsymbol{E}}_v}{\partial \xi} + \frac{\partial \tilde{\boldsymbol{F}}}{\partial \eta} + \frac{\partial \tilde{\boldsymbol{F}}_v}{\partial \eta}\right]^n + A + B + C$$

with

$$\begin{aligned}A = -\Bigg[&\frac{\partial \tilde{\boldsymbol{E}}}{\partial \xi_t}\Delta\xi_t + \frac{\partial \tilde{\boldsymbol{E}}}{\partial \xi_x}\Delta\xi_x + \frac{\partial \tilde{\boldsymbol{E}}}{\partial \xi_r}\Delta\xi_r + \frac{\partial \tilde{\boldsymbol{F}}}{\partial \eta_t}\Delta\eta_t \\ &+ \frac{\partial \tilde{\boldsymbol{F}}}{\partial \eta_x}\Delta\eta_x + \frac{\partial \tilde{\boldsymbol{F}}}{\partial \eta_r}\Delta\eta_r\Bigg]^n\end{aligned} \tag{12}$$

$$B = \frac{1}{J}\tilde{\boldsymbol{U}}^n\Delta J, \qquad C = -\Delta\tilde{t}\tilde{\boldsymbol{G}}^n$$

and $\Delta^n\boldsymbol{Q} = \boldsymbol{Q}^{n+1} - \boldsymbol{Q}^n$ is the time difference.

The ADI method applied to the left-hand side of Eq. (11) gives the following two equations to be solved in each direction:

$$\begin{aligned}\Bigg\{\Gamma + \Delta\tilde{t}\frac{\partial}{\partial \xi}\Bigg[&\frac{\partial \tilde{\boldsymbol{E}}}{\partial \boldsymbol{Q}} + \frac{\partial \tilde{\boldsymbol{E}}_v}{\partial \boldsymbol{Q}_\xi} + \frac{\partial \tilde{\boldsymbol{E}}_v}{\partial \boldsymbol{Q}} \\ &- \left(\frac{\partial \tilde{\boldsymbol{E}}_v}{\partial \boldsymbol{Q}_\xi}\right)_\xi + 0.5\frac{\partial \tilde{\boldsymbol{G}}}{\partial \boldsymbol{Q}}\Bigg]\Bigg\}\Delta^n\boldsymbol{Q}^* = \mathrm{RHS}\end{aligned} \tag{13}$$

$$\begin{aligned}\Bigg\{\Gamma + \Delta\tilde{t}\frac{\partial}{\partial \eta}\Bigg[&\frac{\partial \tilde{\boldsymbol{F}}}{\partial \boldsymbol{Q}} + \frac{\partial \tilde{\boldsymbol{F}}_v}{\partial \boldsymbol{Q}_\eta} + \frac{\partial \tilde{\boldsymbol{F}}_v}{\partial \boldsymbol{Q}} \\ &- \left(\frac{\partial \tilde{\boldsymbol{F}}_v}{\partial \boldsymbol{Q}_\eta}\right)_\eta + 0.5\frac{\partial \tilde{\boldsymbol{G}}}{\partial \boldsymbol{Q}}\Bigg]\Bigg\}\Delta^n\boldsymbol{Q} = \Gamma\Delta^n\boldsymbol{Q}^*\end{aligned} \tag{14}$$

Results and Discussion

A test case for a classical thermodynamics problem of a closed axisymmetric system, e.g., a piston/cylinder assembly, is considered. The gas expansion causes the piston movement. The only force exerted on the piston is caused by the control volume inside the cylinder. Pressurized air ($\gamma = 1.4$), initially at 150 atm, 2000 K, and with an initial volume of 392.7 cm^3, is expanded to 10 times its initial volume. The entire process takes place in less than 0.5 ms by selecting a very light piston mass. Therefore, the expansion can be assumed to be an isentropic process. All of the boundaries are assumed to be adiabatic. The LTCP code[5] was used for this study.

Table 1 shows the solutions obtained by the LTCP code for different cases. These cases demonstrate the significance of the A, B, and $\dot{W}$ components in obtaining a correct numerical solution.

Table 1 Terms included in piston/cylinder test cases

Test case	$\Delta\xi_i$ and $\Delta\eta_i$	$\dot{W}$ in C term	B	A
1	Yes	No	No	No
2	Yes	Yes	No	No
3	Yes	No	Yes	No
4	Yes	No	No	Yes
5	Yes	Yes	Yes	No
6	Yes	Yes	Yes	Yes
7	No	Yes	Yes	Yes

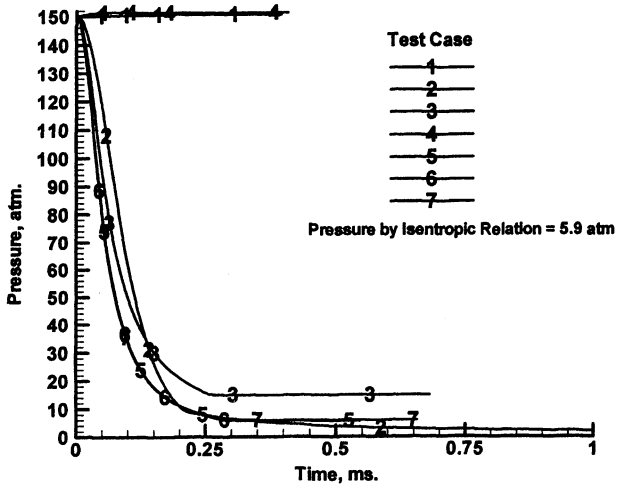

Fig. 1 Pressure variation as a function of time. $P_{initial}$ = 150 atm, $T_{initial}$ = 2000 K.

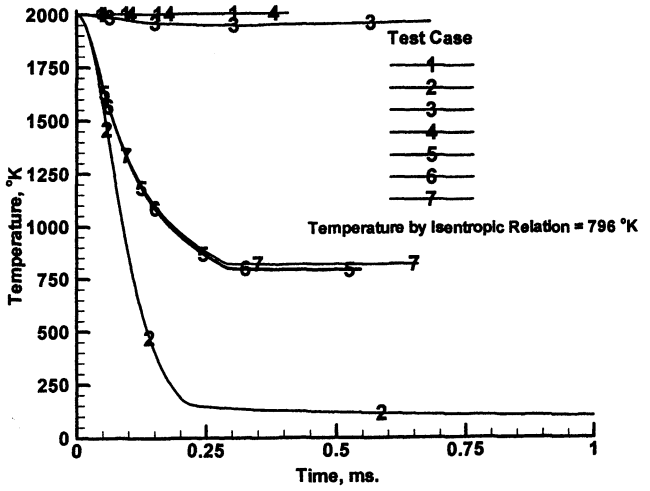

Fig. 2 Temperature variation as a function of time. $P_{initial}$ = 150 atm, $T_{initial}$ = 2000 K.

Figure 1 shows pressure variation as a function of time for all of the test cases. It can be seen that the solution is the most sensitive to inclusion of the $\dot{W}$ and B terms, rather than the A term. Therefore, the terms involving the derivatives of the flux with respect to the transformation metrics can be eliminated for the piston/cylinder assembly problem without any significant loss of accuracy. However, correct results cannot be obtained without the energy source and the time derivative of the Jacobian for the problem at hand. Although, cases 2 and 3 show the same trend in pressure prediction, they yield quite different results for temperature calculations, as is shown in Fig. 2. It appears that the energy equation is quite sensitive to the energy source term. However, without the B term (case 2), the trend of both pressure and temperature are closer to final values obtained by the isentropic relations, yet the values of temperature are quite lower than the isentropic final temperature.

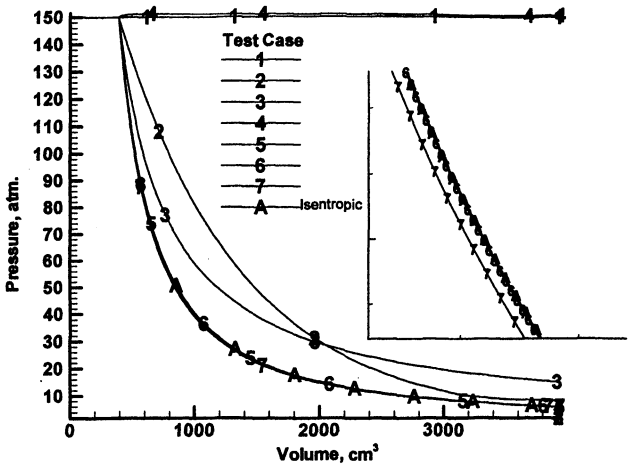

Fig. 3 P–V diagram for test cases and isentropic expansion. $P_{initial}$ = 150 atm, $T_{initial}$ = 2000 K.

Figure 3 demonstrates the P–V diagram for each of the test cases, as compared with the exact isentropic relation. It can clearly be seen that, with the B and $\dot{W}$ terms included in the equations, calculations match the analytical solution extremely well. Test case 7 demonstrates a small sensitivity of the problem at hand to the $\Delta\xi_i$ and $\Delta\eta_i$. This is because the contravariant velocity, which includes a relative Cartesian velocity, is very small or zero, e.g., $(u - x_i)$. Furthermore, a momentum source/sink term should also be included in the momentum equation, for the validation of isentropic relation, which is a result of the energy equation and near-zero gas velocity; this term has been dropped from our formulations.

Conclusions

It was found that, for a closed system with a moving boundary, the inclusion of the boundary work (or power) in the Navier–Stokes equations is required for obtaining correct results. This is also true for open systems with a moving boundary. The time derivative of the Jacobian is another important term for a moving grid that must be included in the discretized equations, to account for the time variation of the effective volume of a cell. For a closed system with initially stagnant fluid, the A term did not show any considerable effect on the solution, because of the very small or near-zero contravariant velocity. We have not yet performed order-of-magnitude analysis of this term for an open system.

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